Assignment 3: Line Integrals and Vector Fields: Solutions

1. (a)

$$\begin{aligned}
& \int_C x^3 + y\,ds \\
= & \int_0^1 ((3t)^3 + (t^3))\sqrt{(3)^2 + (3t^2)^2}\,dt \\
= & \int_0^1 28t^3\sqrt{9 + 9t^4}\,dt \\
= & 84\int_0^1 t^3\sqrt{1 + t^4}\,dt \\
= & 84\left(\frac{(1 + t^4)^{\frac{3}{2}}}{6}\right)_0^1 \\
= & 14(2^{\frac{3}{2}} - 1)
\end{aligned}$$

(b) The equation of the line is $(1 - 3t, 4 + 4t), 0 \leq t \leq 1$. Thus the integral is

$$\begin{aligned}
& \int_0^1 ((1 - 3t)^2 + (4 + 4t)^2)(-3)\,dt \\
= & -3\int_0^1 1 - 6t + 9t^2 + 16 + 32t + 16t^2\,dt \\
= & -3\int_0^1 17 + 26t + 25t^2\,dt \\
= & -3\left(17t + 13t^2 + \frac{25t^3}{3}\right)_0^1 \\
= & -3\left(17 + 13 + \frac{25}{3}\right) \\
= & -115
\end{aligned}$$

(c)

$$\int_C \mathbf{F} \cdot dr$$

$$
\begin{aligned}
&= \int_C P\,dx + Q\,dy + R\,dz \\
&= \int_{-1}^{1} (e^t e^{2t})(e^t) + (e^{-t} + e^{2t})(-e^{-t}) + (e^t)(2e^{2t})\,dt \\
&= \int_{-1}^{1} e^{4t} - e^{-2t} - e^t + 2e^{3t}\,dt \\
&= \left(\frac{e^{4t}}{4} + \frac{e^{-2t}}{2} - e^t + \frac{2e^{3t}}{3}\right)_{-1}^{1} \\
&= \frac{e^4}{4} + \frac{e^{-2}}{2} - e + \frac{2e^3}{3} - \frac{e^{-4}}{4} - \frac{e^2}{2} + e^{-1} - \frac{2e^{-3}}{3}
\end{aligned}
$$

2. Since the path is closed, we can use Green's Theorem. Note that the path is negatively oriented, so the integral becomes

$$-\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

where D is the lower half of the circle $x^2 + y^2 = 4$. Putting in the partial derivatives, this reduces to

$$-\int_D -x\,dA = \int_D x\,dA$$

We now switch to polar co-ordinates to get

$$
\begin{aligned}
&\int_{\pi}^{2\pi} \int_0^2 (r\cos\theta) r\,dr\,d\theta \\
&= \int_{\pi}^{2\pi} \int_0^2 r^2 \cos\theta\,dr\,d\theta \\
&= \frac{8}{3} \int_{\pi}^{2\pi} \cos\theta\,dr\,d\theta \\
&= \frac{8}{3} (\sin\theta)|_{\pi}^{2\pi} \\
&= 0
\end{aligned}
$$

3. (a) Since

$$\frac{\partial Q}{\partial x} = 3y^2 \text{ and } \frac{\partial P}{\partial y} = 3y^2,$$

the vector field is conservative.

(b) If $\bigtriangledown f = \mathbf{F}$, we know

$$\frac{\partial f}{\partial x} = 2x + y^3 \text{ and } \frac{\partial f}{\partial y} = 3xy^2 + 4$$

Integrating the first with respect to x gives

$$f = x^2 + xy^3 + g(y).$$

Differentiating this with respect to y gives

$$\frac{\partial f}{\partial y} = 3x^2y^2 + g'(y)$$

Comparing this with the above expression for $\frac{\partial f}{\partial y}$ gives $g'(y) = 4$, so $g(y) = 4y + C$. So an equation for f is

$$f = x^2 + xy^3 + 4y$$

4. (a) If $\bigtriangledown f = \mathbf{F}$, we know

$$\frac{\partial f}{\partial x} = 8xz \text{ and } \frac{\partial f}{\partial y} = 1 - 6yz^3 \text{ and } \frac{\partial f}{\partial z} = 4x^2 - 9y^2z^2$$

Integrating the first with respect to x gives

$$f = 4x^2z + g(y, z)$$

Differentiating this with respect to y gives

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$$

Comparing this with our above expression gives

$$\frac{\partial g}{\partial y} = 1 - 6yz^3$$

Integrating this with respect to y gives

$$g = y - 3y^2z^3 + h(z)$$

Thus we have

$$f = 4x^2z + y - 3y^2z^3 + h(z)$$

Differentiating this with respect to z gives

$$\frac{\partial f}{\partial z} = 4x^2 - 9y^2z^2 + h'(z)$$

Comparing this with above gives $h'(z) = 0$, so $h(z) = C$. Thus

$$f = 4x^2z + y - 3y^2z^3$$

(b) From (a), we know that $\mathbf{F}$ is conservative. Moreover, the path C is closed. Thus, by the fundamental theorem of line integrals, the integral is 0.